Journal of Mechanical Science and Technology (KSME Int. J.), Vol. 19, No. 6, pp. 1253~1267, 2005 1253

Observability Analysis of Alignment Errors in GPS/INS

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Misalignment can be an important problem in the integration of GPS/INS. Observability analysis of the alignment errors in the integration of low-grade inertial sensors and multi-antenna GPS is presented in this paper. A control-theoretic approach is adopted to study the observability of time-varying error dynamics models. The relationship between vehicle motions and the observability of the errors in the lever arm and relative attitude between GPS antenna array and IMU is given. It is shown that alignment errors can be made observable through maneuvering. The change of acceleration makes the components of the relative attitude error that are orthogonal to the direction of the acceleration change observable. The change of angular velocity makes the components of the lever arm error that are orthogonal to the direction of the angular velocity observable. The motion of constant angular velocity has no influence on the estimation of the lever arm.

Key Words : GPS, INS, Lever Arm, Alignment Error, Observability

Nomenclature

Throughout this paper, the following notations are used :

- ω_{ab}^{c} : Column vector of angular velocity of a frame b relative to a frame a, decomposed in a frame c.
- P^{a} : Position vector decomposed in a frame a.
- V^{a} : Velocity vector decomposed in a frame a.
- R_{a}^{b} : Rotation matrix from a frame a to a frame b.
- Ω_{ab}^{c} : Skew-symmetric cross product matrix of ω_{ab}^{c}.
- I_n : An $n \times n$ identity matrix.
- 0 : A zero matrix with an appropriate dimension.
- $\hat{(\ )}$: Estimated value of ( ).
- $\delta(\ )$: Estimation error of ( ).
- $\dot{(\ )}$: Time derivative of ( ).
- $(\ )^{T}$: Transpose of ( ).
- $|(\ )|$: Absolute value of ( ).
- $\overset{(i)}{A}(t)$: i-th time derivative of a matrix A that is a function of time $\left(=\frac{d^{i}}{(dt)^{i}}A(t)\right)$.
- $(\ )\times(\ )$: Cross product of two vectors.
- $(\ )\cdot(\ )$: Dot (scalar) product of two vectors.

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The navigation frames used in the paper are as follows :

i-frame : Earth-centered inertial (ECI) frame.
e-frame : Earth-centered earth-fixed frame.
t-frame : Earth-fixed tangential frame (east, north, up).
n-frame : Body-fixed navigation frame (north, east, down).
b-frame : Body-frame (forward, right, down).
a-frame : GPS antenna-frame.

1. Introduction

Misalignment can be a serious problem in accurate GPS/INS systems. While the GPS antenna is mounted on the outside of a vehicle, an inertial measurement unit (IMU) is usually placed inside of a vehicle. Thus, the direct measurement of the distance between the GPS antenna and IMU is often quite difficult. The error in the estimated value for the lever arm, the relative position of GPS antennas with respect to the body frame of the inertial sensors, can be of significant magnitude (Bell, 2000-2001). The lever arm error in large vehicles can be much greater than the centimetre-level error in carrier-phase differential GPS (CDGPS) measurement systems. A navigation system using multi-antenna GPS measurements has similar alignment error characteristics. The error in the estimation of the GPS antenna array attitude relative to the inertial sensor frame can be much greater than the attitude measurement error. These alignment errors can increase errors in the estimation of the position, attitude, and inertial sensor biases of vehicles (He and Jianye, 2002 ; Hong et al., 2002).

Estimation of the misalignment between the two sensor systems can be considered as a practical choice in situations when direct measurement of the alignment errors cannot be easily implemented. The precise estimation of the alignment errors requires accurate GPS measurement systems. Thus, the quality of inertial sensors might be considered to be as relatively less important if the GPS measurement update rate is not too slow. This paper investigates the estimation of the alignment errors in the integration of accurate GPS measurement systems with readily available low-cost IMU.

To estimate the alignment errors, observability properties of the GPS/INS system were investigated. The analysis is based upon a null space test of observability matrices for a multi-antenna GPS measurement system (Hong et al., 2002). An INS error dynamics model is expressed in the Earth-centered Earth-fixed (ECEF) frame. Errors in position, velocity, IMU attitude, biases of gyros and accelerometers, GPS antenna lever arm, and the relative attitude of a GPS antenna array were considered in the observability analysis. Among the inertial sensor errors such as biases, scale-factor errors, and alignment errors, biases are most unpredictable and dominant in low-grade sensors. Since the period of testing for the alignment error estimation is relatively short, compared with the time-constant of bias drifts (Goshen-Meskin and Bar-Itzhack, 1992 ; Gebre-Egziabher et al., 1998; Hou and El-Sheimy, 2003), the biases in the inertial sensors are modelled as constant in this paper. It is shown that the time-invariant error dynamics model has six unobservable modes when the position and attitude of a vehicle are measured with a multi-antenna GPS measurement system. Errors in the lever arm and the relative attitude of the antenna array are not observable. Both the error in the relative attitude of the antenna array and the lever arm error contribute to position error. The error in the estimation of the relative attitude of the antenna array also contributes to the error in the estimation of the gyro bias and the attitude of the IMU.

It is shown in this paper that the GPS/INS alignment errors can be made observable through manoeuvring. Based on the observability analysis of time-varying systems, all of the above unobservable modes in the time-invariant error dynamics model are shown to be observable if the vehicle changes both the directions of angular velocity and acceleration. The observability analysis suggests that vehicles should move with various attitudes and accelerations while the measurement data are collected. Changes in acceleration improve the estimation of the relative attitude of the GPS antenna array. The components of the relative attitude error that are orthogonal to the direction of the acceleration change become observable. Changes in the angular velocity decrease the lever arm error. The components of the lever arm error that are orthogonal to the direction of the angular velocity become observable. Similar results on the observability of level arm in the integration of low-cost IMU and single antenna GPS measurement system were found in (Hong et al., 2005). The motion of constant angular velocity does not have any effect on the estimation of the lever arm in the GPS/INS systems in which low-grade inertial sensors are employed. These relationships between the vehicle motions and observability of alignment errors are very consistent with both the simulation and experimental results in (Hong et al., 2004).

The effect of manoeuvring on the improvement of the estimability of INS errors is well known (Baziw and Leondes, 1972; Bar-Itzhack and Porat, 1981; Porat and Bar-Itzhack, 1981). Goshen-Meskin and Bar-Itzhack proposed piecewise constant modelling for the observability analysis of time-varying systems (Goshen-Meskin and Bar-Itzhack, 1992). Using the modelling, they showed that the number of unobservable modes in INS error decreased with change in acceleration (Goshen-Meskin and Bar-Itzhack, 1992). This paper directly studied observability properties of time-varying system. The effects of both translatory and angular motions on the enhancement of the observability for the estimation of alignment errors were given in this paper.

One of the main contributions of this paper is a control-theoretic approach for the observability analysis on general time-varying systems in INS aided by multiantenna GPS measurement system. With this approach, the effects of angular motions as well as translatory motions on the observability of errors in integrated GPS/INS systems can be studied. The second contribution is that the relationships between vehicle motions and the observability of GPS/INS alignment errors are given. The relationships given in this paper are in agreement with the car test results of (Hong et al., 2004) on the estimation of the alignment errors. Thus this paper confirms the validity of the experimental results with a car.

2. Navigation Error Propagation Model

The navigation equations in the ECEF frame are (Wei and Schwarz, 1990; Britting, 1971).

$$\dot{P}^e = V^e \tag{1}$$

$$\dot{V}^e = R_b^e f^b - 2\omega_{ie}^e \times V^e + g^e \tag{2}$$

$$\dot{R}_b^e = R_b^e \Omega_{eb}^b \tag{3}$$

where f^b is the specific force in the body frame and g^e is the gravity in the ECEF frame. The corresponding INS mechanization differential equations are

$$\dot{\hat{P}}^e = \hat{V}^e \tag{4}$$

$$\dot{\hat{V}}^e = \hat{R}_b^e \hat{f}^b - 2\omega_{ie}^e \times \hat{V}^e + \hat{g}^e \tag{5}$$

$$\dot{\hat{R}}_b^e = \hat{R}_b^e \Omega_{eb}^b \tag{6}$$

$$\hat{\omega}_{eb}^b = \hat{\omega}_{ib}^b - \hat{R}_e^b \omega_{ie}^e \tag{7}$$

where $\hat{f}^b$ and $\hat{\omega}_{ib}^b$ are the measurements from accelerometers and gyros, respectively. Let the mechanization errors are modeled as

$$\hat{P}^e = P^e + \delta P \tag{8}$$

$$\hat{V}^e = V^e + \delta V \tag{9}$$

$$\hat{R}_b^e = R_b^e (I_3 + [\gamma \times]) \tag{10}$$

$$\hat{f}^b = f^b + \varepsilon_a + w_a \tag{11}$$

$$\hat{\omega}_{ib}^b = \omega_{ib}^b + \varepsilon_g + w_g \tag{12}$$

where γ is the attitude error, $[\gamma \times]$ is the cross product matrix of γ, ε_a is the accelerometer bias, w_a is the accelerometer noise, ε_g is the gyro bias, and w_g is the gyro noise. Attitude error in INS error analysis has usually been represented in the navigation frame such that $\hat{R}_b^q = (I_3 + [\gamma^q \times]) R_b^q$ where q is the reference frame of INS, such as t or n frame (Wei and Schwarz, 1990; Britting, 1971; Goshen-Meskin and Bar-Itzhack, 1992). The relation between γ^q and γ^b is $\gamma^q = R_b^q \gamma^b$ for small γ^b. Since the major error sources in inertial sensors during run time are biases and the test time for the error estimation is relatively short, error vectors ε_g and ε_a are assumed to be constant biases. Then, the linearized error propagation equations are

$$\delta \dot{P} = \delta V \tag{13}$$

$$\delta \dot{V} = G\delta P - 2\Omega^e_{ie}\delta V - R^e_b F^b \gamma + R^e_b \varepsilon_a + R^e_b w_a \tag{14}$$

$$\dot{\gamma} = -\Omega^b_{ib}\gamma + \varepsilon_g + w_g \tag{15}$$

$$\dot{\varepsilon}_g = 0 \tag{16}$$

$$\dot{\varepsilon}_a = 0 \tag{17}$$

where $G = \frac{\partial g^e}{\partial P^e}$, and F^b, Ω^e_{ie}, and Ω^b_{ib} are the cross product matrices of f^b, ω^e_{ie}, and ω^b_{ib}, respectively. The maximum singular value of G is in the order of 10^{-6} (Nash et al., 1971), the magnitude of ω^e_{ie} is in the order of 10^{-5} (Defense Mapping Agency, World Geodetic System 1984), that of f^b is in the order of 10. The magnitude of δP is in the order of 1, that of δV is in the order of 0.1, and that of γ is in the order of 0.01 in CDGPS. The magnitude of ε_a is in the order of 0.1 and that of ε_g is in the order of 0.001 in very-low grade IMU. Thus, in this paper, the gravity gradient and the angular motion of the earth can be considered less important. Instead of (14) and (15), the following equations are used in the following sections to simplify the observability analysis :

$$\delta \dot{V} = -R^e_b F^b \gamma + R^e_b \varepsilon_a + R^e_b w_a \tag{18}$$

$$\dot{\gamma} = -\Omega^b_{eb}\gamma + \varepsilon_g + w_g \tag{19}$$

3. GPS Measurement Error Model

Consider the measurement system in Fig. 1. In the figure, three GPS antennas are placed on the top surface of a vehicle. IMU is placed inside of the vehicle. Even though three antennas provide three-dimensional attitude, four antennas are usually employed in the commercial products for the improvement of measurement performance. The main GPS antenna, antenna 1, is used for the position measurement. The attitude of the GPS antenna frame is determined with all three GPS antennas. Measurements from GPS receivers can be described as

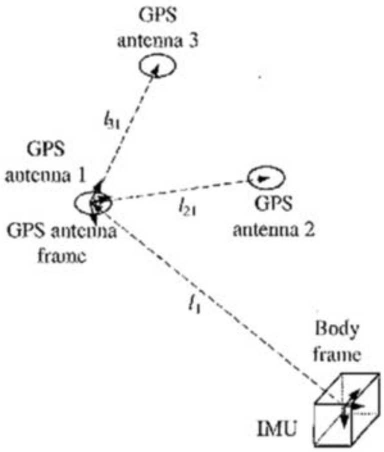

Fig. 1 GPS/INS measurement system

$$P^e_1 = P^e + R^e_b l^b_1 + v_1 \tag{20}$$

$$P^e_{j1} = R^e_a l^a_{j1} + v_{j1},\ j=2,\ 3 \tag{21}$$

where P^e_1 is the measurement for the position of GPS antenna 1, P^e_{j1} is the measurement for the position of GPS antenna j relative to that of GPS antenna 1, l^b_1 is the position of GPS antenna 1 relative to that of IMU decomposed in the body frame, l^a_{j1} is the position of GPS antenna j relative to that of GPS antenna 1 decomposed in the antenna frame, and v_1 and v_{j1} are measurement errors. In this paper, l^a_{21} and l^a_{31} are assumed to be linearly independent so that the attitude of the antenna frame can be determined with the three GPS antennas. Estimations for measurements are given as

$$\hat{P}^e_1 = \hat{P}^e + \hat{R}^e_b \hat{l}^b_1 \tag{22}$$

$$\hat{P}^e_{j1} = \hat{R}^e_b \hat{R}^b_a l^a_{j1} \tag{23}$$

Let the errors in the measurement estimations be defined as

$$\hat{P}^e_1 = P^e_1 + \delta P^e_1 \tag{24}$$

$$\hat{P}^e_{j1} = P^e_{j1} + \delta P^e_{j1} \tag{25}$$

$$\hat{l}^b_1 = l^b_1 + \delta l \tag{26}$$

$$\hat{R}^b_a = (I_3 + [\gamma_a \times]) R^b_a \tag{27}$$

where δl is the lever arm estimation error for antenna 1 and γ_a is the error in the estimation of the antenna frame relative to the body frame.

Then, the linearized measurement estimation errors can be shown as

$$\delta P_1^e = \delta P - R_b^e L_1^b \gamma + R_b^e \delta l - v_1 \quad (28)$$

$$\delta P_{j1}^e = -R_b^e L_{j1}^b (\gamma + \gamma_a) - v_{j1} \quad (29)$$

where L_1^b and L_{j1}^b are the cross product matrices of l_1^b and $R_a^b l_{j1}^a$, respectively.

4. Observability Properties of GPS/INS

In this section, observability analyses are made for both a time-invariant error dynamics model and a time-varying error model. Observability properties are investigated by testing the null space of observability matrices. For the sake of simplicity of analysis, two types of time-varying sys- tems are considered : a system with a time -vary- ing acceleration and a constant attitude, and a system with a constant acceleration and a time- varying attitude.

Before the main part of this section is given, conditions of observability of linear systems used in this paper are introduced to clarify the observability analysis procedure. Consider the linear system :

$$\Sigma : \dot{x}(t) = A(t)x(t)$$
$$y(t) = C(t)x(t)$$

where $A(t)$ and $C(t)$ are respectively the $n \times n$ and $p \times n$ matrices whose entries are continuous functions of t defined over $(-\infty, \infty)$. Define a sequence of $p \times n$ observability matrices $N_0(t)$, $N_1(t)$, $\cdots$, $N_{n-1}(t)$ by the equation

$$N_{k+1}(t) = N_k(t)A(t) + \frac{d}{dt}N_k(t),$$
$$k = 0, 1, 2, \cdots, n-2$$

$$N_0(t) = C(t)$$

Suppose $A(t)$ and $C(t)$ in the system Σ are analytic functions of t. Then, the time-varying system $(A(t), C(t))$ is observable at time t_0 if there exists a finite time $t_1 > t_0$ such that the rank of the matrix

$$\begin{bmatrix} N_0(t_1) \\ N_1(t_1) \\ \vdots \\ N_{n-1}(t_1) \end{bmatrix} \quad (30)$$

is n (Chen, 1984). Suppose $A(t)$ and $C(t)$ in the system Σ are constant. Then, the time-invariant linear system is observable if and only if the rank of the matrix

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is n. If the linear time-invariant system is observable, then it is observable at every initial time, and the determination of the initial state can be achieved in any non-zero time interval (Chen, 1984).

4.1 Time-invariant systems

Suppose R_b^e and F^b in 18 are constant such that $\omega_{eb}^b = 0$. Neglecting the earth's angular motion, this subsection assumes that $\omega_{ib}^b = 0$. Let

$$x = [\delta P^T \ \delta V^T \ \gamma^T \ \varepsilon_g^T \ \varepsilon_a^T \ \delta l^T \ \gamma_a^T]^T \quad (31)$$

$$y = [\delta P_1^{eT} \ \delta P_{21}^{eT} \ \delta P_{31}^{eT}]^T \quad (32)$$

$$A = \begin{bmatrix} 0 & I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_b^e F^b & 0 & R_b^e & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (33)$$

$$C = \begin{bmatrix} I_3 & 0 & -R_b^e L_1^b & 0 & 0 & R_b^e & 0 \\ 0 & 0 & -R_b^e L_{21}^b & 0 & 0 & 0 & -R_b^e L_{21}^b \\ 0 & 0 & -R_b^e L_{31}^b & 0 & 0 & 0 & -R_b^e L_{31}^b \end{bmatrix} \quad (34)$$

Then, the equations of errors for the INS mechanization and measurement estimation are

$$\dot{x} = Ax + w \quad (35)$$

$$y = Cx = v \quad (36)$$

where y is the estimation error for GPS measurements and w and v are the first-order approximation errors. To make observability analysis convenient, consider the following transformation :

$$\bar{x}=T^{-1}x \tag{37}$$

$$\bar{A}=T^{-1}=AT \tag{38}$$

$$\bar{C}=CT \tag{39}$$

with

$$T=\begin{bmatrix} I_3 & 0 & 0 & 0 & 0 & -R_b^e & 0 \\ 0 & I_3 & 0 & R_b^e L_1^b & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & F^b & 0 & I_3 & 0 & 0 \\ 0 & 0 & L_1^b & 0 & 0 & I_3 & 0 \\ 0 & 0 & -I_3 & 0 & 0 & 0 & I_3 \end{bmatrix} \tag{40}$$

$$T^{-1}=\begin{bmatrix} I_3 & 0 & -R_b^e L_1^b & 0 & 0 & R_b^e & 0 \\ 0 & I_3 & 0 & -R_b^e L_1^b & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & -F^b & 0 & I_3 & 0 & 0 \\ 0 & 0 & -L_1^b & 0 & 0 & I_3 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & I_3 \end{bmatrix} \tag{41}$$

Let

$$\bar{x}=[\delta\bar{P}^T \ \delta\bar{V}^T \ \bar{\gamma}^T \ \bar{\varepsilon}_g^T \ \bar{\varepsilon}_a^T \ \delta\bar{l}^T \ \bar{\gamma}_a^T]^T \tag{42}$$

$$\bar{x}_{uo}=\begin{bmatrix} 0 & 0 & I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3 & 0 \end{bmatrix}^T \tag{43}$$

Then, we have the following property :

Property 4.1 : The time-invariant system $(A,\ C)$ has six unobservable modes $\bar{x}_{uo}$.

It can be seen from Property 4.1 that the six unobservable modes are

$$\bar{\gamma}=\gamma \tag{44}$$

$$\delta\bar{l}=\delta l-L_1^b\gamma \tag{45}$$

Due to the unobservable mode γ, γ_a approaches $-\gamma$.

4.2 A system with a time-varying acceleration and a constant attitude

This subsection investigates the effect of change in acceleration on the observability of GPS/INS systems. It is assumed that a vehicle's attitude is fixed and its acceleration is changing. The system matrices for this case are

$$A_f(t)=\begin{bmatrix} 0 & I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_b^e F^b(t) & 0 & R_b^e & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{46}$$

$$C_f=\begin{bmatrix} I_3 & 0 & -R_b^e L_1^b & 0 & 0 & R_b^e & 0 \\ 0 & 0 & -R_b^e L_{21}^b & 0 & 0 & 0 & -R_b^e L_{21}^b \\ 0 & 0 & -R_b^e L_{31}^b & 0 & 0 & 0 & -R_b^e L_{31}^b \end{bmatrix} \tag{47}$$

The corresponding observability matrices are

$$\begin{aligned} N_{r+1}(t) &= N_r(t)A_f(t)+\frac{d}{dt}N_r(t) \\ N_0(t) &= C_f,\ r=0,\ 1,\ \cdots \end{aligned} \tag{48}$$

For the sake of simplicity of analysis, consider the linear transformation

$$M_r(t)\triangleq N_r(t)\,T_f(t),\ r=0,\ 1,\ \cdots,\ n-1 \tag{49}$$

$$\begin{aligned} M_{i+1}(t) &= M_i(t)\tilde{A}_f(t)+\frac{d}{dt}M_i(t),\ \tilde{A}_f(t) \\ &= T_f(t)^{-1}\Big(A_f(t)\,T_f(t)-\frac{d}{dt}(T_f(t))\Big) \end{aligned} \tag{50}$$

where $T_f(t)$ and $T_f(t)^{-1}$ have the same forms as T and T^{-1} in (40) and (41) with time-varying F^b. For this case,

$$M_0(t)=\begin{bmatrix} I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e L_{21}^b \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e L_{31}^b \end{bmatrix} \tag{51}$$

$$\tilde{A}_f(t)=\begin{bmatrix} 0 & I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_b^e & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{dt}F^b(t) & -F^b(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & -L_1^b & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \end{bmatrix} \tag{52}$$

Let $N_r(t)$ and $M_r(t)$ be expressed with 3 by 3 block matrices as follows :

$$N_r(t)=\begin{bmatrix} n_{r(1,1)}(t) & n_{r(1,2)}(t) & \cdots & n_{r(1,7)}(t) \\ n_{r(2,1)}(t) & n_{r(2,2)}(t) & \cdots & n_{r(2,7)}(t) \\ n_{r(3,1)}(t) & n_{r(3,2)}(t) & \cdots & n_{r(3,7)}(t) \end{bmatrix} \quad (53)$$

$$M_r(t)=\begin{bmatrix} m_{r(1,1)}(t) & m_{r(1,2)}(t) & \cdots & m_{r(1,7)}(t) \\ m_{r(2,1)}(t) & m_{r(2,2)}(t) & \cdots & m_{r(2,7)}(t) \\ m_{r(3,1)}(t) & m_{r(3,2)}(t) & \cdots & m_{r(3,7)}(t) \end{bmatrix} \quad (54)$$

Remark 4.1. Since the sixth column blocks in $M_0(t)$ and $\tilde{A}_f(t)$ are zero matrices, we have

$$m_{r(1,6)}(t)=m_{r(2,6)}(t)=m_{r(3,6)}(t)=0, \quad r=0,1,\cdots,n-1 \quad (55)$$

Note 4.1. Suppose the vehicle's attitude is fixed and its acceleration is changing. Then, we have $m_{r(1,3)}(t)=\sum_{j=1}^{r-2} c_{rj}(t)\overset{(j)}{F^b}(t)$, $m_{r(2,3)}(t)=m_{r(3,3)}(t)=0$, $r=3,4,\cdots,n-1$ where $C_{rj}(t)$ is a matrix that is a function of time.

Remark 4.2. It can be seen that $m_{i(j,3)}(t)=0$, $i=0,1,2$, and $j=1,2,3$.

Define unobservable modes such that

$$\bar{x}_{fc}(t)=\begin{bmatrix} 0 & 0 & \left(\frac{d}{dt}f^b(t)\right)^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3 & 0 \end{bmatrix}^T \quad (56)$$

$$\bar{x}_{fv}(t)=[0\ 0\ 0\ 0\ 0\ I_3\ 0]^T \quad (57)$$

Then, we have the following property:

Property 4.2. Suppose R_b^e is constant and F^b is time-varying. If all of the time derivatives of $f^b(t)$ have the same direction, then the time-varying system $(A_f(t), C_f(t))$ has only four unobservable modes, $\bar{x}_{fc}(t)$. Otherwise, $(A_f(t), C_f(t))$ has only three unobservable modes, $\bar{x}_{fv}(t)$.

From Property 4.2 it can be seen that if a vehicle experience acceleration change in a given direction, then the components of attitude error that are perpendicular to the direction of the acceleration change become observable. This result is in agreement with the car test results in (Hong et al., 2004).

4.3 A system with a constant velocity and a time-varying attitude

Next, consider the case in which a vehicle's acceleration is zero and its attitude is changing. In this subsection, it is assumed that the angular velocity of a vehicle is much greater than that of the earth. Thus, the earth is assumed to be motionless such that $\omega_{ib}^b=\omega_{eb}^b$. The system matrices for this case are

$$A_a(t)=\begin{bmatrix} 0 & I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_b^e(t)F^b(t) & 0 & R_b^e(t) & 0 & 0 \\ 0 & 0 & -\Omega_{eb}^b(t) & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (58)$$

$$C_a(t)=\begin{bmatrix} I_3 & 0 & -R_b^e(t)L_1^b & 0 & 0 & R_b^e(t) & 0 \\ 0 & 0 & -R_b^e(t)L_{21}^b & 0 & 0 & 0 & -R_b^e(t)L_{21}^b \\ 0 & 0 & -R_b^e(t)L_{31}^b & 0 & 0 & 0 & -R_b^e(t)L_{31}^b \end{bmatrix} \quad (59)$$

Define the transformation matrix such that

$$T_a(t)=\begin{bmatrix} I_3 & 0 & 0 & 0 & 0 & -R_b^e(t) & 0 \\ 0 & I_3 & 0 & R_b^e(t)L_1^b & 0 & -\overset{(1)}{R_b^e}(t) & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Omega_{eb}^b(t) & I_3 & 0 & 0 & 0 \\ 0 & 0 & F^b(t) & 0 & I_3 & -R_e^b(t)\overset{(2)}{R_b^e}(t) & 0 \\ 0 & 0 & L_1^b & 0 & 0 & I_3 & 0 \\ 0 & 0 & -I_3 & 0 & 0 & 0 & I_3 \end{bmatrix} \quad (60)$$

$$T_a(t)^{-1}=\begin{bmatrix} I_3 & 0 & -R_b^e(t)L_1^b & 0 & 0 & R(t) & 0 \\ 0 & I_3 & \bar{\tau}_{2,3}(t) & -R(t)L_1^b & 0 & \overset{(1)}{R}(t) & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Omega_{eb}^b & I_3 & 0 & 0 & 0 \\ 0 & 0 & \bar{\tau}_{5,3}(t) & 0 & I_3 & R_e^b(t)\overset{(2)}{R_b^e}(t) & 0 \\ 0 & 0 & -L_1^b & 0 & 0 & I_3 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & I_3 \end{bmatrix} \quad (61)$$

where

$$\bar{\tau}_{2,3}(t)=R_b^e(t)L_1^b\Omega_{eb}^b(t)-\overset{(1)}{R_b^e}(t)L_1^b \quad (62)$$

$$\bar{\tau}_{5,3}(t)=-R_e^b(t)\overset{(2)}{R_b^e}(t)L_1^b-F^b(t) \quad (63)$$

Let

$$\mathrm{N}_{i+1}(t)=\mathrm{N}_i(t)A_a(t)+\frac{d}{dt}\mathrm{N}_i(t), \qquad \mathrm{N}_0(t)=C_a(t) \quad (64)$$

where $i=0, 1, \cdots, n-1$, and n is the dimension of the state vector.

As before, consider the linear transformation

$$M_r(t) \triangleq N_r(t)\, T_a(t),\ r=0, 1, \cdots, n-1 \quad (65)$$

$$M_{i+1}(t) = M_i(t)\tilde{A}_a(t) + \frac{d}{dt}M_i(t)$$
$$\tilde{A}_a(t) = T_a(t)^{-1}\left(A_a(t)\,T_a(t) - \frac{d}{dt}(T_a(t))\right) \quad (66)$$

where

$$\tilde{A}_a(t) = \begin{bmatrix} 0 & I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_b^e(t)L_1^b\overset{(1)}{\Omega_{eb}^b}(t) & a_{t24}(t) & R_b^e(t) & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & -\overset{(1)}{\Omega_{eb}^b}(t) & -\Omega_{eb}^b(t) & 0 & 0 & 0 \\ 0 & 0 & -\overset{(1)}{F^b}(t) & a_{t54}(t) & 0 & a_{t56}(t) & 0 \\ 0 & 0 & 0 & -L_1^b & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 \end{bmatrix} \quad (67)$$

with

$$a_{t24}(t) = -2\overset{(1)}{R_b^e}(t)L_1^b + R_b^e(t)L_1^b\Omega_{eb}^b(t) \quad (68)$$

$$a_{t54}(t) = -R_e^b(t)\overset{(2)}{R_b^e}(t)L_1^b - F^b(t) \quad (69)$$

$$a_{t56}(t) = R_e^b(t)\overset{(3)}{R_b^e}(t) + \overset{(1)}{R_e^b}(t)\overset{(2)}{R_b^e}(t) \quad (70)$$

The corresponding observability matrices are

$$M_0(t) = \begin{bmatrix} I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e(t)L_{21}^b \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e(t)L_{31}^b \end{bmatrix} \quad (71)$$

$$M_1(t) = \begin{bmatrix} * & I_3 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & -R_b^e(t)L_{21}^b & 0 & 0 & * \\ * & 0 & 0 & -R_b^e(t)L_{31}^b & 0 & 0 & * \end{bmatrix} \quad (72)$$

$$M_2(t) = \begin{bmatrix} * & * & R_b^e(t)L_1^b\overset{(1)}{\Omega_{eb}^b}(t) & * & R_b^e(t) & 0 & * \\ * & * & R_b^e(t)L_{21}^b\overset{(1)}{\Omega_{eb}^b}(t) & * & 0 & 0 & * \\ * & * & R_b^e(t)L_{31}^b\overset{(1)}{\Omega_{eb}^b}(t) & * & 0 & 0 & * \end{bmatrix} \quad (73)$$

where $*$ is a 3 by 3 matrix. Let $N_r(t)$ and $M_r(t)$ be expressed with smaller block matrices as follows

$$N_r(t) = \begin{bmatrix} v_{r(1,1)}(t) & v_{r(1,2)}(t) & \cdots & v_{r(1,7)}(t) \\ v_{r(2,1)}(t) & v_{r(2,2)}(t) & \cdots & v_{r(2,7)}(t) \\ v_{r(3,1)}(t) & v_{r(3,2)}(t) & \cdots & v_{r(3,7)}(t) \end{bmatrix} \quad (74)$$

$$M_r(t) = \begin{bmatrix} \mu_{r(1,1)}(t) & \mu_{r(1,2)}(t) & \cdots & \mu_{r(1,7)}(t) \\ \mu_{r(2,1)}(t) & \mu_{r(2,2)}(t) & \cdots & \mu_{r(2,7)}(t) \\ \mu_{r(3,1)}(t) & \mu_{r(3,2)}(t) & \cdots & \mu_{r(3,7)}(t) \end{bmatrix} = \begin{bmatrix} \mu_{r,1}(t) \\ \mu_{r,2}(t) \\ \mu_{r,3}(t) \end{bmatrix} \quad (75)$$

where $v_{r(\cdot,\cdot)}(t)$ and $\mu_{r(\cdot,\cdot)}(t)$ are 3 by 3 block matrices, $\mu_{r,j}(t)$ is a 3 by 21 matrix with $r=3, 4, \cdots, n-1$, and $j=1, 2, 3$. Then,

$$\mu_{3(1,3)}(t) = -\left(R_b^e(t)L_1^b\Omega_{eb}^b(t) - 3\overset{(1)}{R_b^e}(t)L_1^b\right)\overset{(1)}{\Omega_{eb}^b}(t) + R_b^e(t)L_1^b\overset{(2)}{\Omega_{eb}^b}(t) - R_b^e(t)\overset{(1)}{F^b}(t)$$
$$\mu_{3(2,3)}(t) = -\left(R_b^e(t)L_{21}^b\Omega_{eb}^b(t) - 3\overset{(1)}{R_b^e}(t)L_{21}^b\right)\overset{(1)}{\Omega_{eb}^b}(t) + R_b^e(t)L_{21}^b\overset{(2)}{\Omega_{eb}^b}(t) \quad (76)$$
$$\mu_{3(3,3)}(t) = -\left(R_b^e(t)L_{31}^b\Omega_{eb}^b(t) - 3\overset{(1)}{R_b^e}(t)L_{31}^b\right)\overset{(1)}{\Omega_{eb}^b}(t) + R_b^e(t)L_{31}^b\overset{(2)}{\Omega_{eb}^b}(t)$$

$$\mu_{3(1,6)}(t) = \overset{(3)}{R_b^e}(t) - \overset{(1)}{R_b^e}(t)R_e^b(t)\overset{(2)}{R_b^e}(t) \quad (77)$$

$$\mu_{r(2,6)}(t) = \mu_{r(3,6)}(t) = 0,\quad r=3, 4, \cdots, n-1 \quad (78)$$

Let

$$X(t) = \Omega_{eb}^b(t)\overset{(1)}{\Omega_{eb}^b}(t) + \overset{(1)}{\Omega_{eb}^b}(t)\Omega_{eb}^b(t) + \overset{(2)}{\Omega_{eb}^b}(t) \quad (79)$$

Then, it follows that

$$\mu_{3(1,6)}(t) = R_b^e(t)X(t) \quad (80)$$

$$\mu_{r(1,6)}(t) = \overset{(r-2)}{R_b^e}(t),\ r=3, 4, \cdots \quad (81)$$

Thus, from (66), (67), and (81), the following relationship can be obtained

$$\mu_{r+1(1,6)}(t) = \overset{(r-2)}{R_b^e}(t)X(t) + \frac{d}{dt}(\mu_{r(1,6)}(t)),\quad r=3, 4, \cdots \quad (82)$$

Note 4.2: For $r=2, 3, \cdots, n-1$, we have the followings:

$$\mu_{r+1(1,3)}(t) = \sum_{j=1}^{r}\left(\alpha_{1,r+1,j}(t)\overset{(j)}{\Omega_{eb}^b}(t)\right) + \sum_{j=1}^{r-1}\left(\beta_{r+1,j}(t)\overset{(j)}{F^b}(t)\right) \quad (83)$$

$$\mu_{r+1(2,3)}(t) = \sum_{j=1}^{r}\left(\alpha_{2,r+1,j}(t)\overset{(j)}{\Omega_{eb}^b}(t)\right) \quad (84)$$

$$\mu_{r+1(3,3)}(t) = \sum_{j=1}^{r}\left(\alpha_{3,r+1,j}(t)\overset{(j)}{\Omega_{eb}^b}(t)\right) \quad (85)$$

where $\alpha_{1,r+1,j}(t)$, $\alpha_{2,r+1,j}(t)$, $\alpha_{3,r+1,j}(t)$, and $\beta_{r+1,j}(t)$ are 3 by 3 matrices that are functions of time. It can be seen that $\beta_{r+1,r-1}(t)=R_b^e(t)$ and $\alpha_{1,r+1,r}(t)=R_b^e(t)L_1^b$ in (83), $\alpha_{2,r+1,r}(t)=R_b^e(t)L_{21}^b$, and $\alpha_{3,r+1,r}(t)=R_b^e(t)L_{31}^b$.

Note 4.3: If $f^b(t)=-g^b(t)$ and $\omega_{te}^b=0$, then

$$\overset{(1)}{f^b}(t)=-\omega_{eb}^b(t)\times f^b(t) \tag{86}$$

Note 4.4: Let $\overset{(1)}{\omega_{eb}^b}(t)$ be constant and be parallel with $\omega_{eb}^b(t)$. Then, only $\delta\bar{l}_u(t)=c_l\overset{(1)}{\omega_{eb}^b}(t)$ satisfies the relationship $\mu_{r(1,6)}(t)\delta\bar{l}_u(t)=0$ for $r=3$, 4, $\cdots$, where c_l is a constant number.

Remark 4.3: If the velocity of a vehicle is not fast, then $\omega_{te}^b\simeq 0$.

The observability conditions for the general angular motion of the vehicle can be quite complicated. In this paper the conditions for relatively simple angular motions are investigated to obtain physical insight on the effects of angular motion of a vehicle on the alignment error estimation. Let

$$\bar{x}_{cp}=\begin{bmatrix}0 & 0 & I_3 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & I_3 & 0\end{bmatrix}^T \tag{87}$$

$$\bar{x}_{cu}=[0\ 0\ 0\ 0\ 0\ I_3\ 0]^T \tag{88}$$

$$\bar{x}_{vp}(t)=\begin{bmatrix}0 & 0 & (\omega_{eb}^b(t))^T & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & (\omega_{eb}^b(t))^T & 0\end{bmatrix}^T \tag{89}$$

$$\bar{x}_{vu}(t)=[0\ 0\ 0\ 0\ 0\ (\omega_{eb}^b(t))^T\ 0]^T \tag{90}$$

Then, we have the following properties:

Property 4.3: Suppose $f^b=-g^b$, $\omega_{et}^e=0$, and ω_{eb}^b is constant. If ω_{eb}^b is parallel with g^b, then the time-varying system $(A_a(t), C_a(t))$ has only six unobservable modes, $\bar{x}_{cp}(t)$. Otherwise, the system $(A_a(t), C_a(t))$ has only three unobservable modes, $\bar{x}_{cu}(t)$.

Property 4.4: Suppose, $f^b=-g^b$, $\omega_{et}^e=0$ and $\overset{(1)}{\omega_{eb}^b}(t)$ is constant and parallel with $\omega_{eb}^b(t)$. If $\omega_{eb}^b(t)$ is parallel with g^b, then the time-varying system $(A_a(t), C_a(t))$ has only two unobservable modes, $\bar{x}_{vp}(t)$. Otherwise, the time-varying system $(A_a(t), C_a(t))$ has only one unobservable mode, $\bar{x}_{vu}(t)$.

Property 4.5: Suppose $f^b=-g^b$, $\omega_{et}^e=0$, $\overset{(1)}{\omega_{eb}^b}$ is constant. Suppose also that both $\omega_{eb}^b(t)$ and $\overset{(1)}{\omega_{eb}^b}$ are not zero-vectors. If $\overset{(1)}{\omega_{eb}^b}$ is not parallel with $\omega_{eb}^b(t)$, then the time-varying system $(A_a(t), C_a(t))$ is observable.

From Property 4.4 it can be seen that if a vehicle experience changes in angular velocity with a given direction, then the components of lever arm error that are perpendicular to the direction of the angular velocity become observable when the magnitude of $L_1^b\gamma$ is much smaller than that of δl. This result is in good agreement with the car test results in (Hong et al., 2004).

Observability analysis can be useful to understand the limitation of measurement systems. If a state is unobservable, the state can not be estimated even with negligibly small sensor noises. However, even though a state is observable, the degree of observability can not be obtained from the observability test. The performance of estimators can usually be tested with covariance simulation with Kalman filter or experiments. The numerical simulation and car test given in (Hong et al., 2004) show that the trends of estimator behavior are in good agreement with the analytical results given in this paper on the observability of the alignment errors. Acceleration changes made the components of the relative attitude error that were perpendicular to the direction of the acceleration change observable. Angular acceleration also made the components of lever arm error that were perpendicular to the direction of the angular acceleration observable.

5. Conclusions

This paper studied the observabilities of alignment errors in the integration of a low-grade IMU with a multi-antenna GPS measurement system. The estimation errors for position, velocity, attitude, biases of inertial sensors, GPS antenna lever arm, and the GPS antenna array attitude with respect to the IMU body frame were considered in the observability analysis.

It was shown that errors in the lever arm and the relative attitude between GPS antenna array and IMU can be made observable through the maneuvering of the vehicle. Acceleration change improves the estimation of the relative attitude of the GPS antenna array. The components of the relative attitude error that are perpendicular to the direction of the acceleration change become observable. Angular acceleration enhances the estimation of the lever arm of the GPS antenna. The components of lever arm error that are perpendicular to the direction of the angular acceleration become observable. However, the motion of constant angular velocity had no influence on the lever arm estimation.

Even though low cost IMU is considered in this paper, attitude GPS measurements are still expensive in common use. An analysis with lower grade attitude GPS receivers including the effect of cycle slips can be a practical research topic for future study.

Acknowledgments

This work was supported by the Korea Science and Engineering Foundation (KOSEF) through the Advanced Ship Engineering Research Center at Pusan Nation University.

Appendix

A.1 Proof of Property 4.1

Note that

$$\bar{C}=\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e L_{21}^b \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e L_{31}^b \end{bmatrix}$$

$$\bar{A}=\begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_b^e & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -F^b & 0 & 0 & 0 \\ 0 & 0 & 0 & -L_1^b & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}$$

Since the third and the sixth column blocks in both $\bar{C}$ and $\bar{A}$ are zero matrices, the same column blocks in $CA^jT=\overline{CA^j}$ are also zero matrices. This completes the proof of Property 4.1.

A.2 Proof of Note 4.1

Let and $\mathcal{M}_r(t)=[m_{r(1,1)}(t) \;\; m_{r(1,2)}(t) \;\; m_{r(1,5)}(t)]$ and

$$\Re=\begin{bmatrix} 0 & I_3 & 0 \\ 0 & 0 & R_b^e \\ 0 & 0 & 0 \end{bmatrix} \tag{91}$$

Then,

$$\begin{gathered}\mathcal{M}_{r+1}(t)=\mathcal{M}_r(t)\Re+\frac{d}{dt}\mathcal{M}_r(t) \\ \mathcal{M}_0(t)=[I_3\ 0\ 0]\end{gathered} \tag{92}$$

$$\begin{gathered}m_{r+1(1,3)}(t)=m_{r(1,5)}(t)\frac{d}{dt}F^b+\frac{d}{dt}(m_{r(1,3)}(t)) \\ m_{0(1,3)}(t)=0\end{gathered} \tag{93}$$

Since $\mathcal{M}_0(t)$ and R are constant matrices, $\mathcal{M}_r(t)$ is also a constant matrix for $r=1, 2, \cdots$. It can be seen that $m_{r(1,5)}$ is a non-zero matrix for $r=2, 3, \cdots$. Thus, $m_{r(1,3)}(t)=\sum_{j=1}^{r-2} c_{rj}(t)\overset{(j)}{F^b}(t)$, $r=3, 4, \cdots, n$, where $c_{rj}(t)$ is a matrix that is a function of time. This completes the proof of Note 4.1.

A.3 Proof of Property 4.2

Note that

$$M_0(t)=\begin{bmatrix} I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e L_{21}^b \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_b^e L_{31}^b \end{bmatrix} \tag{94}$$

$$M_1(t)=\begin{bmatrix} * & I_3 & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & -R_b^e L_{21}^b & 0 & 0 & * \\ * & 0 & 0 & -R_b^e L_{31}^b & 0 & 0 & * \end{bmatrix} \tag{95}$$

$$M_2(t)=\begin{bmatrix} * & * & 0 & * & R_b^e & 0 & * \\ * & * & 0 & * & 0 & 0 & * \\ * & * & 0 & * & 0 & 0 & * \end{bmatrix} \tag{96}$$

$$M_3(t)=\begin{bmatrix} * & * & -R_b^e \overset{(1)}{F^b}(t) & * & * & 0 & * \\ * & * & 0 & * & * & 0 & * \\ * & * & 0 & * & * & 0 & * \end{bmatrix} \tag{97}$$

$$M_4(t)=\begin{bmatrix} * & * & R_b^e \overset{(2)}{F^b}(t) & * & * & 0 & * \\ * & * & 0 & * & * & 0 & * \\ * & * & 0 & * & * & 0 & * \end{bmatrix} \tag{98}$$

Suppose $\bar{x}_{fu}(t)(=[\delta\bar{P}_{fu}^T(t)\ \delta\bar{V}_{fu}^T(t)\ \bar{\gamma}_{fu}^T(t)\ \bar{\varepsilon}_{gfu}^T(t)\ \bar{\varepsilon}_{afu}^T(t)\ \delta\bar{l}_{fu}^T(t)\ \bar{\gamma}_{afu}^T(t)]^T)$ is in the null space of $M_i(t)$, $i=0, 1, \cdots, n$. Then, $M_i(t)\bar{x}_{fu}(t)=0$. $M_0(t)\bar{x}_{fu}(t)=0$ implies that $\delta\bar{P}_{fu}(t)=\bar{\gamma}_{afu}(t)=0$ because l_{21}^b and l_{31}^b are linearly independent and R_b^e is non-singular. $M_1(t)\bar{x}_{fu}(t)=0$ with $\delta\bar{P}_{fu}(t)=\bar{\gamma}_{afu}(t)=0$ implies that $\delta\bar{V}_{fu}(t)=\bar{\varepsilon}_{gfu}(t)=0$. $M_2(t)\bar{x}_{fu}(t)=0$ with $\delta\bar{P}_{fu}(t)=\bar{\gamma}_{afu}(t)=\delta\bar{V}_{fu}(t)=\bar{\varepsilon}_{gfu}(t)=0$ implies that $\delta\bar{\varepsilon}_{afu}(t)=0$. Note that $m_{r(1,3)}(t)=\sum_{j=1}^{r-2}c_{rj}(t)\overset{(j)}{F^b}(t)$ and $m_{r(2,3)}(t)=m_{r(3,3)}(t)=0$ for $r=3, 4, \cdots, n$, and $m_{r(i,6)}(t)=0$ for $r=0, 1, \cdots, n$, $i=1, 2, 3$. Thus, if all of the time derivatives of $f^b(t)$ have the same direction, then $\bar{\gamma}_{fu}(t)=c_f(t)\overset{(1)}{F^b}(t)$, where c_f is a real number that is a function of time. Otherwise, there exists $r>1$ such that $\overset{(1)}{f^b}(t), \overset{(2)}{f^b}(t), \cdots, \overset{(r-1)}{f^b}(t)$ have the same direction and $\overset{(r)}{f^b}(t)$ has a different one. Then, $\overset{(1)}{F^b}(t)\bar{\gamma}_{fu}(t)=\overset{(r)}{F^b}(t)\bar{\gamma}_{fu}(t)=0$. Since $\overset{(1)}{f^b}(t)$ and $\overset{(r)}{f^b}(t)$ have different directions, $\bar{\gamma}_{fu}(t)=0$. In any case, since $m_{r(i,6)}(t)=0$, $r=0, 1, \cdots, n$, $i=1, 23$, it is obvious that $[0\ 0\ 0\ 0\ 0 I_3\ 0]^T$ is in the null space of $M_i(t)$, $i=0, 1, \cdots, n-1$. This completes the proof of Property 4.2.

A.4 Proof of Note 4.2

Note that for $k=1, 2, 3$,

$$v_{r+1(k,3)}(t)=-v_{r(k,2)}R_b^e(t)F^b(t)-v_{r(k,3)}(t)\Omega_{eb}^b(t)+\frac{d}{dt}v_{r(k,3)}(t) \tag{99}$$

$$v_{r+1(k,4)}(t)=v_{r(k,3)}(t)+\frac{d}{dt}v_{r(1,4)}(t) \tag{100}$$

$$v_{r+1(k,5)}(t)=v_{r(k,2)}R_b^e(t)+\frac{d}{dt}v_{r(k,5)}(t) \tag{101}$$

$$v_{r+1(k,6)}(t)=\frac{d}{dt}v_{r(k,6)}(t) \tag{102}$$

$$v_{r+1(k,7)}(t)=\frac{d}{dt}v_{r(k,7)}(t) \tag{103}$$

Thus,

$$\begin{aligned}\mu_{r+(k,3)}(t)&=v_{r+1(k,3)}(t)+v_{r+1(k,4)}(t)\Omega_{eb}^b(t)+v_{r+1(k,5)}(t)F^b(t)\\&\quad+v_{r+1(k,6)}(t)L_1^b-v_{r+1(k,7)}(t)\\&=\frac{d}{dt}(v_{r(k,3)}(t)+v_{r(k,4)}(t)\Omega_{eb}^b(t)+v_{r(k,5)}(t)F^b(t)\\&\quad+v_{r(k,6)}(t)L_1^b-v_{r(k,7)}(t))\\&\quad-v_{r(k,4)}(t)\overset{(1)}{\Omega_{eb}^b}-v_{r(k,5)}(t)\overset{(1)}{F^b}(t)\\&=\frac{d}{dt}(\mu_{r(k,3)}(t))-v_{r(k,4)}(t)\overset{(1)}{\Omega_{eb}^b}(t)-v_{r(k,5)}(t)\overset{(1)}{F^b}(t)\end{aligned} \tag{104}$$

Since $\mu_{0(1,3)}(t)=\mu_{1(1,3)}(t)=0$ and $\mu_{2(1,3)}(t)=-R_b^e(t)L_1^b\overset{(1)}{\Omega_{eb}^b}(t)$, $\mu_{r+1(k,3)}(t)$ is in the form of (83). Note that

$$\begin{aligned}&\begin{bmatrix}v_{r+1(2,1)}(t) & v_{r+1(2,2)}(t) & v_{r+1(2,5)}(t)\\ v_{r+1(3,1)}(t) & v_{r+1(3,2)}(t) & v_{r+1(3,5)}(t)\end{bmatrix}\\&=\begin{bmatrix}v_{r(2,1)}(t) & v_{r(2,2)}(t) & v_{r(2,5)}(t)\\ v_{r(3,1)}(t) & v_{r(3,2)}(t) & v_{r(3,5)}(t)\end{bmatrix}\begin{bmatrix}0 & I_3 & 0\\ 0 & 0 & R_b^e(t)\\ 0 & 0 & 0\end{bmatrix}\\&\quad+\frac{d}{dt}\begin{bmatrix}v_{r(2,1)}(t) & v_{r(2,2)}(t) & v_{r(2,5)}(t)\\ v_{r(3,1)}(t) & v_{r(3,2)}(t) & v_{r(3,5)}(t)\end{bmatrix}\\&\qquad r=0, 1, \cdots, n-1\end{aligned} \tag{105}$$

with

$$\begin{bmatrix}v_{0(2,1)}(t) & v_{0(2,2)}(t) & v_{0(2,5)}(t)\\ v_{0(3,1)}(t) & v_{0(3,2)}(t) & v_{0(3,5)}(t)\end{bmatrix}=0$$

This implies that $v_{r(2,5)}(t)=v_{r(3,5)}(t)=0$, $r=0, 1, \cdots, n-1$. Thus, $\mu_{r(2,3)}(t)$ and $\mu_{r(3,3)}(t)$ have the forms in (84) and (85), respectively. Since $\mu_{2(1,3)}(t)=R_b^e(t)L_1\overset{(1)}{\Omega_{eb}^b}(t)$, $\mu_{2(2,3)}(t)=R_b^e(t)L_{21}\overset{(1)}{\Omega_{eb}^b}(t)$, $\mu_{2(3,3)}(t)=R_b^e(t)L_{31}\overset{(1)}{\Omega_{eb}^b}(t)$, $v_{2(1,4)}(t)=R_b^e(t)L_1\Omega_{eb}^b(t)-2\overset{(1)}{R_b^e}(t)L_1$, $v_{2(2,4)}(t)=R_b^e(t)L_{21}\Omega_{eb}^b(t)-2\overset{(1)}{R_b^e}(t)L_{21}$, $v_{2(3,4)}(t)=R_b^e(t)L_{31}\Omega_{eb}^b(t)-2\overset{(1)}{R_b^e}(t)L_{31}$, and $v_{2(1,5)}(t)=R_b^e(t)$, then, $\beta_{r+1,r+1}(t)=R_b^e(t)$, $\alpha_{1,r+1,r}(t)=R_b^e(t)L_1^b$, $\alpha_{2,r+1,r}(t)=R_b^e(t)L_{21}^b$, and $\alpha_{3,r+1,r}(t)=R_b^e(t)L_{31}^b$. This completes the proof of Note 4.3.

A.5 Proof of Note 4.3

Note that $f^b(t)=R_t^b(t)f^t=-R_t^b(t)g^t$ where g^t is the gravity vector in the tangential frame. If the vehicle does not move very fast, it can be assumed that g^t is constant. Thus,

$$\begin{aligned}\frac{d}{dt}f^b(t)&=\frac{d}{dt}(R_t^b(t))f^t\\&=\Omega_{bt}^b(t)R_t^b(t)f^t\\&=-\Omega_{tb}^b(t)f^b(t)\end{aligned} \tag{106}$$

Since $\omega_{et}^b=0$, $\omega_{tb}^b(t)=\omega_{eb}^b(t)$. Thus, $\frac{d}{dt}f^b(t)=-\omega_{eb}^b(t)\times f^b(t)$. This completes the proof of Note 4.3.

A.6 Proof of Note 4.4

Since $\overset{(1)}{\omega_{eb}^b}$ is parallel with $\omega_{eb}^b(t)$, $\overset{(1)}{\omega_{eb}^b}=k_v(t)\,\omega_{eb}^b(t)$ where $k_v(t)$ is a scalar-valued function of time. Considering that $\overset{(1)}{\omega_{eb}^b}$ is time-invariant, we have the following relationships:

$$\mathrm{X}(t)=\Omega_{eb}^b(t)\overset{(1)}{\Omega_{eb}^b}+\overset{(1)}{\Omega_{eb}^b}\Omega_{eb}^b(t)=2k_v(t)\,(\Omega_{eb}^b(t))^2 \tag{107}$$

$$\overset{(1)}{\mathrm{X}}(t)=2\overset{(1)}{\Omega_{eb}^b}\overset{(1)}{\Omega_{eb}^b}=2(k_v(t))^2(\Omega_{eb}^b(t))^2 \tag{108}$$

$$\overset{(r)}{\mathrm{X}}(t)=0,\ r=2,3,\cdots \tag{109}$$

Thus, from (82),

$$\mu_{3(1,6)}(t)=R_b^e(t)\mathrm{X}(t) \tag{110}$$

$$\mu_{4(1,6)}(t)=2\overset{(1)}{R_b^e}(t)\mathrm{X}(t)+R_b^e(t)\overset{(1)}{\mathrm{X}}(t) \tag{111}$$

$$\mu_{r(1,6)}(t)=(r-2)\overset{(r-3)}{R_b^e}(t)\mathrm{X}(t)+\frac{(r-2)(r-3)}{2}\overset{(r-4)}{R_b^e}(t)\overset{(1)}{\mathrm{X}}(t),\quad r=5,6,\cdots \tag{112}$$

From these relations, it can be seen that only $\delta\bar{l}_u(t)=c_l\overset{(1)}{\omega_{eb}^b}$ can satisfy the relationship $\mu_{r(1,6)}(t)\,\delta\bar{l}_u(t)=0$, $r=3,4,\cdots$. This completes the proof of Note 4.4.

A.7 Proof of Property 4.3

Suppose $x_u(t)$ is an unobservable mode of the system $(A_a(t), C_a(t))$. Let $\bar{x}_u(t)=T_t^{-1}(t)\,x_u(t)$. Then

$$\mu_{r,j}(t)\bar{x}_u(t)=0,\ r=0,1,\cdots,n-1,\ j=1,2,3 \tag{113}$$

Let $\bar{x}_u(t)=[(\delta\bar{P}_u(t))^T\ (\delta\bar{V}_u(t))^T\ (\bar{\gamma}_u(t))^T\ (\bar{\varepsilon}_{gu}(t))^T\ (\bar{\varepsilon}_{au}(t))^T\ (\delta\bar{l}_u(t))^T\ (\bar{\gamma}_{au}(t))^T]^T$. Then $\mu_{0,1}(t)\bar{x}_u(t)=0$ implies that $\delta\bar{P}_u(t)=0$. Since l_{21} and l_{31} are linearly independent and $R_b^e(t)$ is non-singular, $\mu_{0,2}(t)\bar{x}_u(t)=\mu_{0,3}(t)\bar{x}_u(t)=0$ implies that $\bar{\gamma}_{au}(t)=0$. $\mu_{1,1}(t)\bar{x}_u(t)=0$ with $\delta\bar{P}_u(t)=\bar{\gamma}_{au}(t)=0$ implies that $\delta\bar{V}_u(t)=0$. $\mu_{1,2}(t)\bar{x}_u(t)=\mu_{1,3}(t)\bar{x}_u(t)=0$ with $\delta\bar{P}_u(t)=\bar{\gamma}_{au}(t)=0$ implies that $\bar{\varepsilon}_{gu}(t)=0$, for the same reason we used when it was proven that $\bar{\gamma}_{au}(t)=0$. Since ω_{eb}^b is constant, $\mathrm{X}(t)$ and $\mu_{r(1,6)}(t)$ in (79) and (82) are zero matrices for $r=3,4,\cdots$. Hence, $\mu_{r(j,u)}(t)$ is a zero matrix for $j=1,2,3$, $r=0,1,\cdots$. Thus, $\delta\bar{l}_u(t)$ can be any vector. If ω_{eb}^b is parallel with g^b, then it can be seen from Note 4.3 that $\overset{(r)}{f^b}(t)$ is a zero vector for $r=1,2,\cdots$. Thus, from Note 4.2 it can be seen that $\mu_{r(j,3)}(t)$ is a zero matrix for $r=2,3,\cdots$, $j=1,2,3$. Hence, $\mu_{r(j,3)}(t)$ is a zero matrix for $r=0,1,\cdots$, $j=1,2,3$. Therefore, $\bar{\gamma}_u(t)$ can be any vector. If a constant vector ω_{eb}^b is not parallel with g^b, then $\overset{(1)}{f^b}(t)$ and $\overset{(2)}{f^b}(t)$ are linearly independent non-zero vectors. Thus, $\mu_{3,1}(t)\bar{x}_u(t)=\mu_{4,1}(t)\bar{x}_u(t)=0$ with $\delta\bar{P}_u(t)=\delta\bar{V}_u(t)=\bar{\varepsilon}_{gu}(t)=\bar{\varepsilon}_{au}(t)=\bar{\gamma}_{au}(t)=0$ implies that $\overset{(u)}{f^b}(t)\times\bar{\gamma}_u(t)=\overset{(2)}{f^b}(t)\times\bar{\gamma}_u(t)=0$. This means that $\bar{\gamma}_u(t)=0$, because $\overset{(1)}{f^b}(t)$ is not parallel with $\overset{(2)}{f^b}(t)$. This completes the proof of Property 4.3.

A.8 Proof of Property 4.4

Suppose $x_u(t)$ is an unobservable mode for $(A_a(t), C_a(t))$ and $\bar{x}_u(t)=T_a(t)x_u(t)$. Let $\bar{x}_u(t)=[(\delta\bar{P}_u(t))^T\ (\delta\bar{V}_u(t))^T\ (\bar{\gamma}_u(t))^T\ (\bar{\varepsilon}_{gu}(t))^T\ (\bar{\varepsilon}_{au}(t))^T\ (\delta\bar{l}_u(t))^T\ (\bar{\gamma}_{au}(t))^T]^T$. It can be shown that $\delta\bar{P}_u(t)=\delta\bar{V}_u(t)=\bar{\gamma}_{au}(t)=\bar{\varepsilon}_{gu}(t)=0$ for the same reasoning as shown in Property 4.3. Since $\overset{(1)}{\omega_{eb}^b}$ is a non-zero vector and l_{21}^b is not parallel with l_{31}^b, $\mu_{2,2}(t)\bar{x}_u(t)=\mu_{2,3}(t)\bar{x}_u(t)=0$ with $\delta\bar{P}_u(t)=\delta\bar{V}_u(t)=\bar{\gamma}_{au}(t)=\bar{\varepsilon}_{gu}(t)=0$ implies that $\bar{\gamma}_u(t)=c_\gamma\overset{(1)}{\omega_{eb}^b}$ where c_γ is a constant number. $\mu_{21}(t)\bar{x}_u(t)=0$ with $\delta\bar{P}_u(t)=\delta\bar{V}_u(t)=\bar{\gamma}_{au}(t)=\bar{\varepsilon}_{gu}(t)=0$ and $\bar{\gamma}_u(t)=c_\gamma\overset{(1)}{\omega_{eb}^b}$ implies that $R_b^e(t)\bar{\varepsilon}_{au}(t)=0$. This means $\bar{\varepsilon}_{au}(t)=0$, because $R_b^e(t)$ is non-singular. From Note 4.2, it can be seen that $\mu_{r(2,3)}(t)\overset{(1)}{\omega_{eb}^b}=\mu_{r(3,3)}(t)\overset{(1)}{\omega_{eb}^b}=0$ for $r=3,4,\cdots$, because $\overset{(1)}{\omega_{eb}^b}$ is constant. If $\omega_{eb}^b(t)$ is parallel with g^b, then $\overset{(r)}{f^b}(t)=0$ for $r=1,2,\cdots$. Hence, $\mu_{r,j}(t)\bar{x}_u(t)=0$ for $r=0,1,\cdots$, $j=1,2,3$ with $\delta\bar{P}_u(t)=\delta\bar{V}_u(t)=\bar{\gamma}_{au}(t)=\bar{\varepsilon}_{au}(t)=\bar{\varepsilon}_{gu}(t)=0$ and $\bar{\gamma}_u(t)=c_\gamma\overset{(1)}{\omega_{eb}^b}$ implies that $\mu_{r(1,6)}(t)\,\delta\bar{l}_u(t)=0$ for $r=0,1,\cdots$.

Since $\overset{(1)}{\omega}{}^b_{eb}$ is constant, it can be seen from Note 4.4 that $\delta \bar{l}_u(t) = c_l \overset{(1)}{\omega}{}^b_{eb}$. If $\omega^b_{eb}(t)$ is not parallel with g^b, then $\mu_{3,1}(t)\bar{x}_u(t) = \mu_{4,1}(t)\bar{x}_u(t) = 0$ with $\delta \bar{P}_u(t) = \delta \bar{V}_u(t) = \bar{\gamma}_{au}(t) = \bar{\varepsilon}_{au}(t) = \bar{\varepsilon}_{gu}(t) = 0$ and $\bar{\gamma}_u(t) = c_\gamma \overset{(1)}{\omega}{}^b_{eb}$. This implies that

$$-c_\gamma R^e_b(t) \overset{(1)}{F}{}^b(t) \overset{(1)}{\omega}{}^b_{eb} + R^e_b(t) \mathrm{X}(t) \delta \bar{l}_u(t) = 0 \quad (114)$$

$$-c_\gamma R^e_b(t) \overset{(2)}{F}{}^b(t) \overset{(1)}{\omega}{}^b_{eb} + R^e_b(t) \overset{(1)}{\mathrm{X}}(t) \delta \bar{l}_u(t) = 0 \quad (115)$$

Let $\overset{(1)}{\omega}{}^b_{eb} = k_v(t)\,\omega^b_{eb}(t)$, where $k_v(t)$ is a scalar-valued function of time. Then, with constant $\overset{(1)}{\omega}{}^b_{eb}$, it follows that $\mathrm{X}(t) = 2k_v(t)(\Omega^b_{eb})^2$, $\overset{(1)}{\mathrm{X}}(t) = 2(k_v(t))^2(\Omega^b_{eb})^2$, $\overset{(1)}{f}{}^b(t) = -\Omega^b_{eb}(t) f^b(t)$, and $\overset{(2)}{f}{}^b(t) = (-k_v(t)\Omega^b_{eb}(t) + (\Omega^b_{eb}(t))^2) f^b(t)$. Note that $\overset{(1)}{\mathrm{X}}(t) = k_v(t)\mathrm{X}(t)$ and $\overset{(1)}{f}{}^b(t)$ is not parallel with $\overset{(2)}{f}{}^b(t)$. Thus, $c_\gamma = 0$ and $\delta \bar{l}_u(t) = c_l(t)\,\omega^b_{eb}(t)$. This completes the proof of Property 4.4.

A.9 Proof of Property 4.5

Suppose $x_u(t)$ is an unobservable mode for $(A_a(t), C_a(t))$ and $\bar{x}_u(t) = T_a(t)x_u(t)$. Let $\bar{x}_u(t) = [(\delta \bar{P}_u(t))^T \ (\delta \bar{V}_u(t))^T \ (\bar{\gamma}_u(t))^T \ (\bar{\varepsilon}_{gu}(t))^T \ (\bar{\varepsilon}_{au}(t))^T \ (\delta \bar{l}_u(t))^T \ (\bar{\gamma}_{au}(t))^T]^T$. Obviously, $\mu_{r,j}(t)\bar{x}_u(t) = 0$ for $r = 0, 1, 2, j = 1, 2, 3$ implies the following condition

$$\begin{aligned} \delta \bar{P}_u(t) = \delta \bar{V}_u(t) = \bar{\gamma}_{au}(t) = \bar{\varepsilon}_{gu}(t) = 0 \\ \bar{\gamma}_u(t) = c_\gamma(t) \overset{(1)}{\omega}{}^b_{ab} \end{aligned} \quad (116)$$

where $c_\gamma(t)$ is a scalar-valued function of time. $\mu_{3,1}(t)\bar{x}_u(t) = 0$ with the condition implies that

$$-c_\gamma(t) \overset{(1)}{F}{}^b(t) \overset{(1)}{\omega}{}^b_{eb} + \mathrm{X}(t) \delta \bar{l}_u(t) = 0 \quad (117)$$

$\mu_{4,1}(t)\bar{x}_u(t) = 0$ with (116) implies that

$$\begin{aligned} &c_\gamma(t)\Big(-2\overset{(1)}{R}{}^e_b(t) \overset{(1)}{F}{}^b(t) - R^e_b(t) \overset{(2)}{F}{}^b(t)\Big) \overset{(1)}{\omega}{}^b_{eb} \\ &+\Big(2\overset{(1)}{R}{}^e_b(t)\mathrm{X}(t) + R^e_b(t)\overset{(1)}{\mathrm{X}}(t)\Big)\delta \bar{l}_u(t) = 0 \end{aligned} \quad (118)$$

Considering (117), (118) implies that

$$-c_\gamma(t) \overset{(2)}{F}{}^b(t) \overset{(1)}{\omega}{}^b_{eb} + \overset{(1)}{\mathrm{X}}(t) \delta \bar{l}_u(t) = 0 \quad (119)$$

$\mu_{5,1}(t)\bar{x}_u(t) = 0$ with (116) implies that

$$\begin{aligned} &c_\gamma(t)\Big(-3\overset{(2)}{R}{}^e_b(t) \overset{(1)}{F}{}^b(t) - 3\overset{(1)}{R}{}^e_b(t) \overset{(2)}{F}{}^b(t) - R^e_b(t) \overset{(3)}{F}{}^b(t)\Big) \overset{(1)}{\omega}{}^e_{eb} \\ &+\Big(3\overset{(2)}{R}{}^e_b(t)\mathrm{X}(t) + 3\overset{(1)}{R}{}^e_b(t)\overset{(1)}{\mathrm{X}}(t)\Big)\delta \bar{l}_u(t) = 0 \end{aligned} \quad (120)$$

Considering (117) and (119), (120) implies that

$$c_\gamma(t) \overset{(3)}{F}{}^b(t) \overset{(1)}{\omega}{}^b_{eb} = 0 \quad (121)$$

$\mu_{6,1}(t)\bar{x}_u(t) = 0$ with (116) implies that

$$\begin{aligned} &c_\gamma(t)\Big(-4\overset{(3)}{R}{}^e_b(t) \overset{(1)}{F}{}^b(t) - 6\overset{(2)}{R}{}^e_b(t) \overset{(2)}{F}{}^b(t) \\ &-4\overset{(1)}{R}{}^e_b(t) \overset{(3)}{F}{}^b(t) - R^e_b(t) \overset{(4)}{F}{}^b(t)\Big) \overset{(1)}{\omega}{}^b_{eb} \\ &+\Big(4\overset{(3)}{R}{}^e_b(t)\mathrm{X}(t) + 6\overset{(2)}{R}{}^e_b(t)\overset{(1)}{\mathrm{X}}(t)\Big)\delta \bar{l}_u(t) = 0 \end{aligned} \quad (122)$$

Considering (117), (119), and (121), (122) implies that

$$c_\gamma(t) \overset{(4)}{F}{}^b(t) \overset{(1)}{\omega}{}^b_{eb} = 0 \quad (123)$$

Since $\omega^b_{eb}(t)$ is not parallel with $\overset{(1)}{\omega}{}^b_{eb}$, the three vectors $\omega^b_{eb}(t)$, $\overset{(1)}{\omega}{}^b_{eb}$, and $\overset{(1)}{\omega}{}^b_{eb} \times \omega^b_{eb}(t)$ are linearly independent. Hence, the specific force can be decomposed with the vectors such that

$$\begin{aligned} f^b(t) = k_1(t)\,\omega^b_{eb}(t) + k_2(t) \overset{(1)}{\omega}{}^b_{eb} \\ + k_3(t) \overset{(1)}{\omega}{}^b_{eb} \times \omega^b_{eb}(t) \end{aligned} \quad (124)$$

where $k_1(t)$, $k_2(t)$, and $k_3(t)$ are scalar-valued functions of time. Considering that

$$\overset{(3)}{f}{}^b(t) = \Big(2\overset{(1)}{\Omega}{}^b_{eb}\Omega^b_{eb}(t) + \Omega^b_{eb}(t)\overset{(1)}{\Omega}{}^b_{eb} - (\Omega^b_{eb}(t))^3\Big) f^b(t) \quad (125)$$

$$\begin{aligned} \overset{(4)}{f}{}^b = c_\gamma(t)\Big(3\overset{(1)}{\Omega}{}^b_{eb}\overset{(1)}{\Omega}{}^b_{eb} - 3\overset{(1)}{\Omega}{}^b_{eb}\Omega^b_{eb}(t)\Omega^b_{eb}(t)\Big) f^b(t) \\ -\Omega^b_{eb}(t) \overset{(3)}{f}{}^b(t) \end{aligned} \quad (126)$$

(121) and (123) implies that

$$\begin{aligned} &\Big(2\overset{(1)}{\Omega}{}^b_{eb}\Omega^b_{eb}(t) + \Omega^b_{eb}(t)\overset{(1)}{\Omega}{}^b_{eb} - (\Omega^b_{eb}(t))^3\Big) f^b(t) \\ &-c_1(t)\overset{(1)}{\omega}{}^b_{eb} = 0 \end{aligned} \quad (127)$$

$$\begin{aligned} &c_\gamma(t)\Big(3\overset{(1)}{\Omega}{}^b_{eb}\overset{(1)}{\Omega}{}^b_{eb} - 3\overset{(1)}{\Omega}{}^b_{eb}\Omega^b_{eb}(t)\Omega^b_{eb}(t)\Big) f^b(t) \\ &-c_1(t)\Omega^b_{eb}(t)\overset{(1)}{\omega}{}^b_{eb} - c_2(t)\overset{(1)}{\omega}{}^b_{eb} = 0 \end{aligned} \quad (128)$$

where $c_1(t)$ and $c_2(t)$ are scalar-valued functions of time. Substituting (124) into these equations, it follows that

$$\begin{aligned}&\Big(-c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_1(t)+2c_7(t)\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_2(t)\\&-c_7(t)|\omega^b_{eb}(t)|^2\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_3(t)\Big)\omega^b_{eb}(t)\\&+\Big(c_7(t)|\omega^b_{eb}(t)|^2k_1(t)-2c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_2(t)\\&+c_7(t)|\omega^b_{eb}(t)|^4k_3(t)-c_1(t)\Big)\overset{(1)}{\omega}{}^b_{eb}\\&+\Big(-c_7(t)|\omega^b_{eb}(t)|^2k_2(t)\\&-3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_3(t)\Big)\overset{(1)}{\omega}{}^b_{eb}\times\omega^b_{eb}(t)=0\end{aligned}\tag{129}$$

$$\begin{aligned}&\Big(-3c_7(t)\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_1(t)-3c_7(t)|\omega^b_{eb}(t)|^2\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_3(t)\Big)\omega^b_{eb}(t)\\&+\Big(3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_1(t)\\&+3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)|\omega^b_{eb}(t)|^2k_3(t)-c_2(t)\Big)\overset{(1)}{\omega}{}^b_{eb}\\&+\Big(-3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_2(t)\\&-3c_7(t)\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_3(t)+c_1(t)\Big)\overset{(1)}{\omega}{}^b_{eb}\times\omega^b_{eb}(t)=0\end{aligned}\tag{130}$$

These mean

$$\begin{aligned}&-c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_1(t)+2c_7(t)\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_2(t)\\&-c_7(t)|\omega^b_{eb}(t)|^2\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_3(t)=0\end{aligned}\tag{131}$$

$$\begin{aligned}&c_7(t)|\omega^b_{eb}(t)|^2k_1(t)-2c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_2(t)\\&+c_7(t)|\omega^b_{eb}(t)|^4k_3(t)-c_1(t)=0\end{aligned}\tag{132}$$

$$\begin{aligned}&-c_7(t)|\omega^b_{eb}(t)|^2k_2(t)\\&-3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_3(t)=0\end{aligned}\tag{133}$$

$$\begin{aligned}&-3c_7(t)\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_1(t)\\&-3c_7(t)|\omega^b_{eb}(t)|^2\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_3(t)=0\end{aligned}\tag{134}$$

$$\begin{aligned}&3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_1(t)\\&+3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)|\omega^b_{eb}(t)|^2k_3(t)-c_2(t)=0\end{aligned}\tag{135}$$

$$\begin{aligned}&-3c_7(t)\Big(\overset{(1)}{\omega}{}^b_{eb}\cdot\omega^b_{eb}(t)\Big)k_2(t)\\&-3c_7(t)\Big|\overset{(1)}{\omega}{}^b_{eb}\Big|^2k_3(t)+c_1(t)=0\end{aligned}\tag{136}$$

From (131) and (134), it follows that $c_7(t)k_1(t)=-c_7(t)|\omega^b_{eb}(t)|^2k_3(t)$ and $c_7(t)k_2(t)=0$. Applying these relationships to (132) and (136), it follows that $c_1(t)=0$ and $c_7(t)k_3(t)=0$. Thus, $c_7(t)k_1(t)=c_7(t)k_2(t)=c_7(t)k_3(t)=0$. If $c_7(t)\neq 0$, then $k_1(t)=k_2(t)=k_3(t)=0$. This is impossible because $f^b(t)=-R^b_t(t)g^b\neq 0$. Thus,

$$c_7(t)=0\tag{137}$$

Applying this relation to (119), we have $\overset{(1)}{X}(t)\,\delta\bar{l}_u(t)=0$. Since $\overset{(1)}{X}(t)=2\overset{(1)}{\Omega}{}^b_{eb}\overset{(1)}{\Omega}{}^b_{eb}$ for constant $\overset{(1)}{\omega}{}^b_{eb}$, only the following relation satisfies (119)

$$\delta\bar{l}_u(t)=c_l(t)\,\overset{(1)}{\omega}{}^b_{eb}\tag{138}$$

where $c_l(t)$ is a scalar-valued function of time. If (138) and (137) are applied to (117), it follows that $c_l(t)X(t)\overset{(1)}{\omega}{}^b_{eb}=0$. Since $X(t)=\overset{(1)}{\Omega}{}^b_{eb}\Omega^b_{eb}(t)+\Omega^b_{eb}(t)\overset{(1)}{\Omega}{}^b_{eb}$ for constant $\overset{(1)}{\omega}{}^b_{eb}$, it follows that $c_l(t)\overset{(1)}{\omega}{}^b_{eb}\times\omega^b_{eb}(t)\times\overset{(1)}{\omega}{}^b_{eb}=0$. Since $\overset{(1)}{\omega}{}^b_{eb}$ is not parallel with ω^b_{eb}, $c_l(t)=0$. Thus, it follows that $x_u(t)=\bar{x}_u(t)=0$. Therefore, the time-varying system $(A_a(t),C_a(t))$ is observable. This completes the proof of Property 4.5.

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